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Results on the propagator of a periodically kicked harmonic oscillator

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I. INTRODUCTION

Non-autonomous dynamical systems which are characterized by a classical Hamiltonian of the form

$$H(p,q,t) := H_0(p,q) + V(q) \sum_{n=-\infty}^{+\infty} \delta(n - t/\tau) \ , \quad (p,q) \in \Gamma \ , \quad \tau > 0 \ , \qquad (1)$$

are of particular interest for various reasons. The Hamiltonian (1) describes an one-dimensional system which evolves regularly in the phase space Γ during time intervals $t \in (n\tau - \tau, n\tau)$ according to the autonomous Hamiltonian $H_0(p,q)$ followed by instant kicks at times $t = n\tau$, which are described by a kicking potential $V(q)$. Due to this time-dependent perturbation the system (1) becomes non-integrable and typically shows chaotic behavior.[1] Actually, the equations of motion derived from Hamiltonian (1) lead to an area-preserving map which also appears as Poincaré surface of sections of autonomous systems in more than one dimension.

Another reason for investigating dynamical systems of the form (1) is that they are well suited for studying the corresponding quantum dynamics. The properties of quantum systems showing chaos in the classical limit have now been of interest for more that 10 years.[2] Most of the interest has been concentrated on the kicked free particle on the unit circle[3] and the Euclidean line[4] where $H_0(p,q)$ is the free Hamiltonian on the phase space $\Gamma = \mathrm{R} \times S^1$ and $\Gamma = \mathrm{R}^2$, respectively. A recent review has been given by Casati and Molinari.[5]

In this contribution we will consider the dynamical system of a periodically kicked harmonic oscillator, that is, we set in (1)

$$H_0(p,q) := \frac{p^2}{2M} + \frac{M\Omega^2}{2} q^2 \ , \quad (p,q) \in \Gamma := \mathrm{R}^2 \ . \qquad (2)$$

In the above $M > 0$ and $\Omega > 0$ are the mass and the frequency of the harmonic oscillator, respectively, which is periodically kicked in time according to (1). For a quadratic kicking potential $V(q)$ such a system has been studied by Blümel, Meir and Smilanski.[6] Here we will not specify the kicking potential.

In section II we will discuss the classical properties of this model showing regular and irregular behavior. Then, in section III we are considering the corresponding quantum dynamics. For the resonance case, where the oscillator frequency Ω is an integer or half-integer multiple of the kicking frequency $2\pi/\tau$, the corresponding classical motion becomes regular. We derive an explicit closed-form expression for the quantum propagator at these resonances. The calculation for this propagator utilizes a Feynman path-integral-like expression. It appears that only the classical path is contributing to the path integral. We also give an explicit result for a quadratic kicking potential via the Van Vleck formula. For a general kicking potential and no resonance we perform a quasi-classical evaluation for the path integral of the quantum propagator.

II. THE CLASSICAL SYSTEM

The classical dynamics of the kicked harmonic oscillator in phase space is described by the Hamilton equations following from Hamiltonian (1) with (2). These equations take the form of difference equations which describe an area-preserving map in the phase space $\Gamma = \mathrm{R}^2$:

$$\begin{aligned} q_{n+1} &= q_n \cos(\Omega\tau) + p_n \sin(\Omega\tau)/\Omega M \ , \\ p_{n+1} &= p_n \cos(\Omega\tau) - q_n \Omega M \sin(\Omega\tau) - \tau V'(q_{n+1}) \ . \end{aligned} \qquad (3)$$

In the above $V'(q) := \partial V(q)/\partial q$. Furthermore, we have set $q_n := q(n\tau + 0)$ and $p_n := p(n\tau + 0)$ which are the position and the momentum of the harmonic oscillator right after the n-th kick. Given an initial point $(p_0, q_0) \in \Gamma$, the classical motion in phase space is uniquely determined by the map (3). That is, $p_n = p_n(p_0, q_0)$ and $q_n = q_n(p_0, q_0)$, which are in general not explicitly known. However, for the resonance case where $\Omega\tau$ is a multiple of π the equations (3) allow for an explicit solution.

For the case of integer resonances, that is, $\Omega\tau/2\pi = k$, $k \in \mathrm{N}$, the solution reads

$$q_n = q_0 \ , \quad p_n = p_0 - n\tau V'(q_0) \ . \qquad (4)$$

In the case of half-odd-integer resonances, $\Omega\tau/2\pi = k + \frac{1}{2}$, $k \in \mathrm{N}_0$, we find

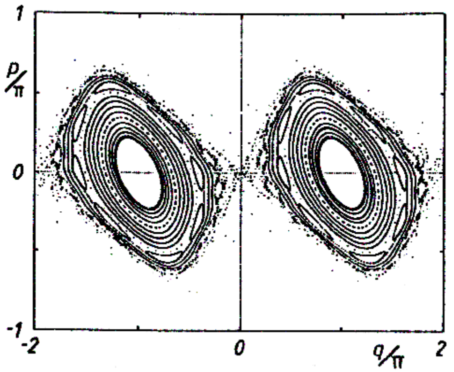

FIG. 1. Phase space portrait of the kicked harmonic oscillator (1) with kicking potential $V(q) = \kappa\cos(q)$. The parameters are $\Omega = 10^{-5}\pi$, $\kappa = 1$.

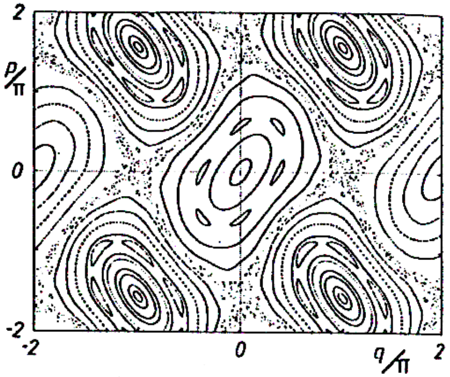

FIG. 2. Same as figure 1 with parameters $\Omega = \pi/2$ and $\kappa = 2$.

$$
\begin{aligned}
q_n &= (-1)^n q_0 \,, \\
p_{2m} &= p_0 + m\tau\,[V'(-q_0) - V'(q_0)] \qquad\qquad \text{for} \quad n = 2m\,, \\
p_{2m+1} &= p_0 - m\tau\,[V'(-q_0) - V'(q_0)] - \tau V'(q_0) \quad \text{for} \quad n = 2m+1\,.
\end{aligned} \tag{5}
$$

By elimination of the momentum degree of freedom in (3) one derives the following second-order difference equation describing the classical motion in q-space:

$$
q_{n+1} = 2\cos(\Omega\tau)\,q_n - (\tau/\Omega M)\sin(\Omega\tau)\,V'(q_n) - q_{n-1}. \tag{6}
$$

Given an initial pair (q_0, q_1) the position after, say, N kicks is uniquely determined, that is $q_N = q_N(q_0, q_1)$. On the other hand, for a given initial point q_0 and a given end point q_N there may be several solutions of the boundary condition

$$
q_N = q_N(q_0, q_1^\alpha). \tag{7}
$$

Here we have introduced an index α in order to enumerate the distinct solutions of (7). The set of points $\{q_n^\alpha\}_{n=0,1,\dots,N}$ forms a classical "path" which starts at $t = 0$ in $q_0^\alpha \equiv q_0$ and ends in $q_N^\alpha \equiv q_N$ at $t = N\tau$. With each such path α we may associate a classical action

$$
S_{\mathrm{cl}}^\alpha(q_N, q_0) := \sum_{n=1}^{N} S(q_n^\alpha, q_{n-1}^\alpha) \tag{8}
$$

where the one-kick action is defined by

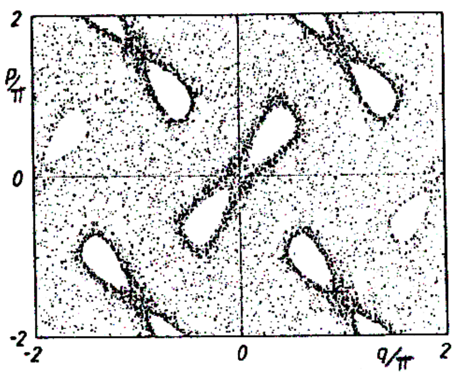

FIG. 3. Same as figure 1 with parameters $\Omega = \pi/2$ and $\kappa = 4$.

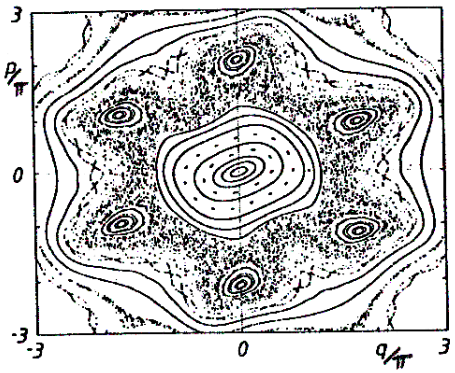

FIG. 4. Same as figure 1 with parameters $\Omega = 1$ and $\kappa = 0.75$.

$$
S(x, x') := \frac{M\Omega}{2\sin(\Omega\tau)}\left[(x^2 + x'^2)\cos(\Omega\tau) - 2xx'\right] - \tau V(x). \tag{9}
$$

Note that from this one-kick action the map (3) can be rederived using the relations $p_n = -\partial S(q_{n+1}, q_n)/\partial q_n$ and $p_{n+1} = \partial S(q_{n+1}, q_n)/\partial q_{n+1}$.

We have performed a numerical iteration of the map (3) for a kicking potential of the form $V(q) := \kappa\cos(q)$ and for various values of the frequency Ω and kicking strength κ. The units have been set to $M = \tau = 1$. Figure 1 shows a phase-space portrait for $\Omega = 10^{-5}\pi$ and $\kappa = 1$. As we expect for such a small Ω, we obtain the phase-space picture of the standard map. Figure 2 and 3 are produced using a frequency $\Omega = \pi/2$ and a kicking strength $\kappa = 2$ and 4, respectively. These figures clearly show the transition from regular to chaotic motion. Figure 4 demonstrates that even for small kicking strengths, here $\kappa = 0.75$, the system may exhibit chaotic motion if the frequency Ω, here $\Omega = 1$, is small enough.

III. THE QUANTUM PROPAGATOR

The quantum dynamics of a dynamical system of the form (1) can be studied by considering the one-kick propagator

$$
\hat{U} := \exp\left\{-\frac{\mathrm{i}}{\hbar}\tau V(\hat{q})\right\}\exp\left\{-\frac{\mathrm{i}}{\hbar}\tau H_0(\hat{p}, \hat{q})\right\}. \tag{10}
$$

This operator describes the time evolution of system (1) for one period τ. That is, the system evolves between two successive kicks with the unperturbed Hamiltonian $H_0(\hat{p}, \hat{q})$ followed by an instant kick described by the operator $V(\hat{q})$. Here, $\hat{p}$ and

$\hat{q}$ denote the usual momentum and position operator with commutation relation $\hat{p}\hat{q} - \hat{q}\hat{p} = \hbar/\mathrm{i}$.

In order to understand the long-time behavior one has to find a simple and possibly closed-form expression for the N-th power of $\hat{U}$. For this reason, we will work in the q-representation where $\hat{U}^N$ takes a Feynman path-integral-like form:

$$\langle x_N|\hat{U}^N|x_0\rangle = \int_{-\infty}^{+\infty} \mathrm{d}x_{N-1} \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_1 \, \langle x_N|\hat{U}|x_{N-1}\rangle \cdots \langle x_1|\hat{U}|x_0\rangle. \tag{11}$$

Another way of studying the quantum dynamics of the kicked system is to introduce the so-called quasi-energy operator $\hat{E}$ which is defined by

$$\exp\{-(\mathrm{i}/\hbar)\hat{E}\tau\} := \hat{U}. \tag{12}$$

Knowledge of the eigenvalues and eigenstates of $\hat{E}$ provides all information about the dynamics we are looking for.

In the following we will study the path integral (11) for the case of a kicked harmonic oscillator where $H_0(\hat{p}, \hat{q})$ is given via (2). Whenever possible we will also give closed-form expressions for the quasi-energy operator $\hat{E}$ and its eigenvalues and eigenstates.

A. Exact solution at resonances

In section II we have shown that the classical dynamical system (1) with (2) becomes integrable, independent of the explicit form of the kicking potential $V(q)$, whenever there is resonance between the kicking frequency $2\pi/\tau$ and the oscillator frequency Ω. Therefore, it is natural to expect also an integrable quantum model in these resonance cases.

For integer resonances, $\Omega\tau/2\pi = k$, $k \in \mathrm{N}$, one has for the regular harmonic motion between two successive kicks

$$\langle x|\exp\{-(\mathrm{i}/\hbar)\tau H_0(\hat{p}, \hat{q})\}|x'\rangle = \exp\{-\mathrm{i}\pi k\}\delta(x - x'). \tag{13}$$

Hence, the path integral (11) takes the form

$$\langle x_N|\hat{U}^N|x_0\rangle = \int_{-\infty}^{+\infty} \mathrm{d}x_{N-1} \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_1 \, \mathrm{e}^{-\mathrm{i}\pi Nk} \prod_{n=1}^{N} [\mathrm{e}^{-(\mathrm{i}/\hbar)\tau V(x_n)} \delta(x_n - x_{n-1})] \tag{14}$$

and is easily calculated due to the delta functions. Note that only a single path, which is actually the classical path (4), does contribute to the above path integral. The final result for the propagator reads

$$\langle q_N|\hat{U}^N|q_0\rangle = \exp\left\{-\frac{\mathrm{i}}{\hbar}\left[V(q_0) + \frac{\hbar\Omega}{2}\right]\tau N\right\}\delta(q_N - q_0). \tag{15}$$

From this expression one also obtains a closed-form expression for the quasi-energy operator (12):

$$\hat{E} = V(\hat{q}) + \hbar\Omega/2. \tag{16}$$

Obviously, this operator has a purely continuous spectrum being that of the kicking potential shifted by the zero-point energy of the harmonic oscillator. The corresponding (generalized) eigenstates are those of the position operator $\hat{q}$.

For the half-odd-integer resonances $\Omega\tau/2\pi = k + \frac{1}{2}$, $k \in \mathrm{N}_0$, we have

$$\langle x|\exp\{-(\mathrm{i}/\hbar)\tau H_0(\hat{p}, \hat{q})\}|x'\rangle = \exp\{-\mathrm{i}\pi(k + 1/2)\}\delta(x + x') \tag{17}$$

which leads to the path integral

$$\langle x_N|\hat{U}^N|x_0\rangle = \int_{-\infty}^{+\infty} \mathrm{d}x_{N-1} \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_1 \, \mathrm{e}^{-\mathrm{i}\pi N(k+1/2)} \prod_{n=1}^{N} \left[\mathrm{e}^{-(\mathrm{i}/\hbar)\tau V(x_n)} \delta(x_n + x_{n-1})\right]. \tag{18}$$

Again, only the classical path, here the one given in (5), does contribute:

$$\langle q_N|\hat{U}^N|q_0\rangle = \exp\left\{-\frac{\mathrm{i}}{\hbar}\left[\frac{\hbar\Omega}{2} + \frac{1}{N}\sum_{n=1}^{N} V((-1)^n q_0)\right] N\tau\right\} \delta(q_N - (-1)^N q_0). \tag{19}$$

Here, in general, we cannot give a closed-form expression for the quasi-energy operator. However, for a kicking potential with even parity, that is $[V(\hat{q}), \hat{P}] = 0$ where $\hat{P}$ denotes the parity operator defined by $\hat{P}|x\rangle := |-x\rangle$, we also arrive at the result (16) if we neglect a possibly remaining $\hat{P}$, that is $\hat{U}^N = \hat{P}^N \exp\{-(\mathrm{i}/\hbar)\hat{E}N\tau\}$. For an arbitrary kicking potential the one-kick propagator is of the form

$$\hat{U} = \exp\{-(\mathrm{i}/\hbar)(V(\hat{q}) + \hbar\Omega/2)\tau\}\hat{P} = \hat{P}\exp\{-(\mathrm{i}/\hbar)(V(-\hat{q}) + \hbar\Omega/2)\tau\} \tag{20}$$

The resonance cases which have led to integrable systems in the classical considerations of section II also lead to explicitly solvable path integrals. It is an interesting fact that only a single path, namely the classical one, does contribute. There are no quantum fluctuations which have to be taken into account. Even the quantum propagator for the regular harmonic motion between two successive kicks does not depend on $\hbar$ at resonance. Note that the results in this section are obtained for an arbitrary kicking potential $V(q)$.

B. Exact solution for quadratic kicking potential

There exists also a special kicking potential where we can calculate the path integral (11) exactly even for off-resonance cases. This is, no surprise, a harmonic kicking potential

$$V(q) := (M/2)\omega^2 q^2 \quad , \quad \omega^2\tau^2 \in \mathrm{R} \ . \tag{21}$$

As the Hamiltonian is quadratic, the quantum propagator can directly be calculated via the quasi–classical Van Vleck formula,

$$\langle q_N|\hat{U}^N|q_0\rangle = \sqrt{\frac{\mathrm{i}}{2\pi\hbar}\,\frac{\partial^2 S_{\mathrm{cl}}(q_N,q_0)}{\partial q_N \partial q_0}}\,\exp\left\{\frac{\mathrm{i}}{\hbar}\,S_{\mathrm{cl}}(q_N,q_0)\right\}, \tag{22}$$

which becomes exact for such systems.[7] Hence, we have to find an explicit form for the classical action which, indeed, can be found by noting that the solution of the equation of motion (6) for given boundary condition (7) is

$$q_n = \frac{q_N - q_0\mathrm{e}^{-\mathrm{i}\varphi N}}{\mathrm{e}^{\mathrm{i}\varphi N}-\mathrm{e}^{-\mathrm{i}\varphi N}}\,\mathrm{e}^{\mathrm{i}\varphi n} + \frac{q_0\mathrm{e}^{\mathrm{i}\varphi N} - q_N}{\mathrm{e}^{\mathrm{i}\varphi N}-\mathrm{e}^{-\mathrm{i}\varphi N}}\,\mathrm{e}^{-\mathrm{i}\varphi n}. \tag{23}$$

Here we have set

$$\cos\varphi := \cos(\Omega\tau) - \frac{\tau\omega^2}{2\Omega}\sin(\Omega\tau). \tag{24}$$

Note that the angle φ takes the values $\varphi \in (0,\pi)$, $\varphi = \mathrm{i}\lambda$ or $\varphi = \pi + \mathrm{i}\lambda$, where $\lambda > 0$. The particular values $\varphi = 0$ and π correspond to resonance cases and will not be considered here. The classical action associated with the path given by (23) reads

$$S_{\mathrm{cl}}(q_N,q_0) = \frac{M\Omega}{2\sin(\Omega\tau)}\,\frac{\sin\varphi}{\sin(N\varphi)}\left[\left(q_N^2+q_0^2\right)\cos(N\varphi) - 2q_Nq_0\right] + \frac{M}{4}\omega^2\tau^2\left(q_0^2-q_N^2\right) \tag{25}$$

and leads to the propagator

$$\langle q_N|\hat{U}^N|q_0\rangle = \sqrt{\frac{M\Omega}{2\pi\mathrm{i}\hbar\sin(\Omega\tau)}\,\frac{\sin\varphi}{\sin(N\varphi)}}\,\exp\left\{\frac{\mathrm{i}}{\hbar}\,S_{\mathrm{cl}}(q_N,q_0)\right\}. \tag{26}$$

This propagator is, up to a unitary transformation, identical in form with that of an ordinary continuous–time harmonic oscillator with frequency $(\arcsin(\sin\varphi)/\tau)$ and mass $\left(M\frac{\Omega\tau}{\sin(\Omega\tau)}\frac{\sin\varphi}{\arcsin(\sin\varphi)}\right)$ which are always real. Therefore, we immediately obtain the corresponding quasi–energy operator

$$\mathrm{e}^{\mathrm{i}(M\omega^2\tau^2/4)\hat{q}^2}\,\hat{E}\,\mathrm{e}^{-\mathrm{i}(M\omega^2\tau^2/4)\hat{q}^2} = \frac{\arcsin(\sin\varphi)}{\sin(\varphi)}\,\frac{\sin(\Omega\tau)}{\Omega\tau}\left[\frac{\hat{p}^2}{2M} + \frac{M}{2}\left(\frac{\Omega\tau\sin\varphi}{\sin(\Omega\tau)}\right)^2\hat{q}^2\right] \tag{27}$$

which coincides with the earlier result of Blümel, Meir and Smilanski[6] using an operator method.

In the above there has not yet been taken into account the possible appearance of Maslov–like phases. In the next section, where we perform an explicit quasi–classical approximation, we will also take care of these phases.

C. Quasi–classical approximation to the quantum propagator

In contrast to the special cases discussed so far the path integral (11) cannot be calculated explicitly for a general kicking potential when there is no resonance. However, it is possible to perform the integrations approximately in the quasi–classical limit $\hbar \to 0$. Such a calculation for the kicked free particle has been made by Tabor.[8] For some improvements and corrections on Tabor's results see ref. [9].

The general form of the one–kick propagator (11) in the q–representation reads

$$\langle x|\hat{U}|x'\rangle = \left(\frac{M\Omega}{2\pi\hbar|\sin(\Omega\tau)|}\right)^{1/2}\exp\left\{-\mathrm{i}\frac{\pi}{4} - \mathrm{i}\frac{\pi}{2}\left[\frac{\Omega\tau}{\pi}\right]\right\}\exp\left\{\frac{\mathrm{i}}{\hbar}\,S(x,x')\right\} \tag{28}$$

with the one–kick action given by (9). In the above $[z]$ stands for the largest integer less or equal to z. The path integral (11) takes the form

$$\begin{aligned}\langle x_N|\hat{U}^N|x_0\rangle = &\left(\frac{M\Omega}{2\pi\hbar|\sin(\Omega\tau)|}\right)^{N/2}\exp\left\{-\mathrm{i}\frac{\pi}{4}N - \mathrm{i}\frac{\pi}{2}\left[\frac{\Omega\tau}{\pi}\right]N\right\}\\ &\times\int_{-\infty}^{+\infty}\mathrm{d}x_{N-1}\cdots\int_{-\infty}^{+\infty}\mathrm{d}x_1\,\exp\left\{\frac{\mathrm{i}}{\hbar}\,S_N(x_N,\ldots,x_0)\right\}\end{aligned} \tag{29}$$

where the action along the path $\{x_n\}_{n=0,\ldots,N}$ is defined by

$$S_N(x_N,\ldots,x_0) := \sum_{n=1}^{N}\left(\frac{M\Omega}{2\sin(\Omega\tau)}\left[(x_n^2+x_{n-1}^2)\cos(\Omega\tau) - 2x_nx_{n-1}\right] - \tau V(x_n)\right). \tag{30}$$

The simplest approach for a quasi–classical evaluation of the path integral (29) is to expand the action (30) about the classical paths $\{q_n^\alpha\}_{n=0,\ldots,N}$ to second order, performing the remaining Fresnel integrals and summing up these contributions.[7] For this, we set $\xi_n^\alpha := x_n - q_n^\alpha$ and expand the action (30) up to second order in ξ. Neglecting the higher–order terms we find ($V''(q) := \partial^2 V(q)/\partial q^2$)

$$\begin{aligned}S_N(x_N,\ldots,x_0) \approx\, &S_{\mathrm{cl}}^\alpha(q_N,q_0)\\ &+\sum_{n=1}^{N}\frac{M\Omega}{2\sin(\Omega\tau)}\left[\left(\xi_n^{\alpha\,2}+\xi_{n-1}^{\alpha\,2}\right)\cos(\Omega\tau) - 2\xi_n^\alpha\xi_{n-1}^\alpha\right] - \frac{\tau V''(q_n^\alpha)}{2}\,\xi_n^{\alpha\,2}.\end{aligned} \tag{31}$$

The contribution of the quadratic fluctuations about the classical path α to the propagator is of the form $F_N^\alpha\exp\{(\mathrm{i}/\hbar)S_{\mathrm{cl}}^\alpha(q_N,q_0)\}$ where we have defined

$$F_N^\alpha := \sqrt{\frac{M\Omega}{2\pi\hbar|\sin(\Omega\tau)|}}\,\exp\left\{-\mathrm{i}\frac{\pi}{4}-\mathrm{i}\frac{\pi}{2}\left[\frac{\Omega\tau}{\pi}\right]N\right\}\left(\frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{\pi}}\right)^{1}\int_{-\infty}^{+\infty}\mathrm{d}z_{N-1}\cdots\int_{-\infty}^{+\infty}\mathrm{d}z_1$$
$$\times\exp\left\{\mathrm{i}\sigma\sum_{n=1}^{N}\left[(z_n^2+z_{n-1}^2)\cos(\Omega\tau)-2z_nz_{n-1}-\frac{\tau\sin(\Omega\tau)}{M\Omega}V''(q_n^\alpha)z_n^2\right]\right\} \tag{32}$$

with

$$z_n := \left(\frac{M\Omega}{2\hbar|\sin(\Omega\tau)|}\right)^{1/2}\xi_n^\alpha\,,\quad \sigma := \frac{\sin(\Omega\tau)}{|\sin(\Omega\tau)|}\in\{-1,+1\}\,. \tag{33}$$

The integration may be performed by diagonalizing the quadratic form in the exponent (note that $z_0 = z_N = 0$) and using the Fresnel–integral formula

$$\frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\mathrm{d}z\,\mathrm{e}^{\mathrm{i}\sigma\lambda z^2} = \frac{1}{\sqrt{|\lambda|}}\begin{cases}1 & \text{for }\ \sigma\lambda>0\\ \mathrm{e}^{-\mathrm{i}\pi/2} & \text{for }\ \sigma\lambda<0\end{cases}. \tag{34}$$

The result can be written as follows

$$F_N^\alpha = \sqrt{\frac{M\Omega}{2\pi\hbar|\sin(\Omega\tau)|\,|\det G_N^\alpha|}}\,\exp\left\{-\mathrm{i}\frac{\pi}{4}-\mathrm{i}\frac{\pi}{2}\left[\frac{\Omega\tau}{\pi}\right]N-\mathrm{i}\frac{\pi}{2}\nu_\alpha\right\}. \tag{35}$$

Here G_N^α is the tridiagonal $(N-1)\times(N-1)$-matrix

$$G_N^\alpha := \begin{pmatrix} d_1^\alpha & -1 & 0 & \cdots & & 0\\ -1 & d_2^\alpha & -1 & 0 & \cdots & 0\\ \vdots & & & & & \vdots\\ 0 & \cdots & 0 & -1 & d_{N-2}^\alpha & -1\\ 0 & & \cdots & 0 & -1 & d_{N-1}^\alpha\end{pmatrix},\quad d_n^\alpha := 2\cos(\Omega\tau)-\frac{\tau\sin(\Omega\tau)}{M\Omega}V''(q_n^\alpha), \tag{36}$$

and ν_α is a Maslov-like index. This index is equal to the number of negative eigenvalues of G_N^α for $[\Omega\tau/\pi]$ even ($\sigma = 1$) and equals the number of positive eigenvalues of G_N^α if $[\Omega\tau/\pi]$ is odd ($\sigma = -1$), respectively.

The complete propagator in the quasi–classical approximation is obtained by summing up the contributions of all classical paths connecting q_0 and q_N by N time steps τ:

$$\langle q_N|\hat{U}^N|q_0\rangle \approx \sum_\alpha\sqrt{\frac{M\Omega}{2\pi\hbar|\sin(\Omega\tau)\det G_N^\alpha|}}\,\exp\left\{-\mathrm{i}\frac{\pi}{2}\left(\left[\frac{\Omega\tau}{\pi}\right]N+\nu_\alpha+\frac{1}{2}\right)\right\}$$
$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}S_{\mathrm{cl}}^\alpha(q_N,q_0)\right\}. \tag{37}$$

The contribution of a path α to the transition amplitude (37) consists of a phase and an amplitude. In the phase there appear beside the classical action Maslov-like indices. The phase $[\Omega\tau/\pi]N$ is due to regular harmonic motion between the N kicks. The additional phase ν_α is a consequence of the periodic kicks. The amplitude in (37) contains the determinant of the matrix (36). For the particular case of the quadratic kicking potential (21) one finds $\det G_N^\alpha = \sin(N\varphi)/\sin\varphi$. The additional phase reads $\nu_\alpha = [N\varphi/\pi]$ for $\varphi \in (0,\pi)$, $\nu_\alpha = 0$ for $\varphi = \mathrm{i}\lambda$, and $\nu_\alpha = (-1)^{N-1}$ for $\varphi = \pi + \mathrm{i}\lambda$, respectively. A detailed discussion on the properties of the determinant of matrices of the form (36) can be found in the previous work.[9]

IV. CONCLUDING REMARKS

In this paper we have presented results on the quantum propagator of a harmonic oscillator which is periodically kicked by a force described via the kicking potential $V(q)$. Classically this system shows regular as well as chaotic behavior. In particular, for resonance between the kicking frequency and the harmonic oscillations the system becomes integrable. For this case we have been able to calculate the Feynman path–integral–like expression for the corresponding quantum propagator exactly. It turns out that only a single path, namely the classical one, does contribute to the path sum. We have also presented a closed–form expression for the associated quasi–energy operator for arbitrary kicking potential. This operator is usually hard to find in closed form.

In the general off–resonance case, where no explicit calculation is possible, we have performed a quasi–classical evaluation for the path integral. In particular, we have taken care about the possible appearance of Maslov–like phases in the propagator. In the usual cases of continuous–time systems and the kicked free particle these indices are always the number of negative eigenvalues of some matrix similar to (36).[7,9] Here it can happen that these indices are the number of positive eigenvalues.

By construction, the quasi–classical approach is exact for quadratic kicking potentials. Hence, one rediscovers the result obtained via the Van Vleck formula but including also the correct phases. In contrast to the resonance cases where only the classical path is contributing, for the quadratic kicking potential, also the quadratic fluctuations about the classical path have to be taken into account in order to result in the exact propagator.

Finally, we like to mention that the next–order corrections in $\hbar$ to the quasi–classical approximation (37) can be performed in analogy to the treatment for the kicked free particle.[10]

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